Semi-classical limit of the generalized second lowest eigenvalue of Dirichlet Laplacians on small domains in path spaces

Shigeki Aida
Mathematical Institute
Tohoku University, Sendai, 980-8578, JAPAN
e-mail: aida@math.tohoku.ac.jp

Abstract

Let M be a complete Riemannian manifold. Let $P_{x,y}(M)$ be the space of continuous paths on M with fixed starting point x and ending point y. Assume that x and y is close enough such that the minimal geodesic $c_{x,y}$ between x and y is unique. Let $-L_\lambda$ be the Ornstein-Uhlenbeck operator with the Dirichlet boundary condition on a small neighborhood of the geodesic $c_{x,y}$ in $P_{x,y}(M)$. The underlying measure $\bar{\nu}^\lambda_{x,y}$ of the L^2-space is the normalized probability measure of the restriction of the pinned Brownian motion measure on the neighborhood of $c_{x,y}$ and λ^{-1} is the variance parameter of the Brownian motion. We show that the generalized second lowest eigenvalue of $-L_\lambda$ divided by λ converges to the lowest eigenvalue of the Hessian of the energy function of the H^1-paths at $c_{x,y}$ under the small variance limit (semi-classical limit) $\lambda \to \infty$.

1 Introduction

Let (M,g) be an n-dimensional complete Riemannian manifold. Let $x,y \in M$. Let $\nu^\lambda_{x,y}$ be the pinned Brownian motion measure on the pinned path space $P_{x,y}(M) = C([0,1] \to M \mid \gamma(0) = x, \gamma(1) = y)$, where λ is a positive parameter such that the transition probability of the Brownian motion is given by $p(t/\lambda, x, y)$, where $p(t,x,y)$ denotes the heat kernel of the L^2-semigroup $e^{t\Delta/2}$ on $L^2(M,dx)$. Here dx denotes the Riemannian volume and Δ is the Laplace-Beltrami operator. The parameter λ coincides with the inverse number of the variance parameter of the Brownian motion on a Euclidean space which is obtained by the stochastic-development of the Brownian motion on M to the tangent space T_xM at the starting point x. Let D_0 be the H-derivative on $P_{x,y}(M)$ and consider the Dirichlet form in $L^2(P_{x,y}(M), d\nu^\lambda_{x,y})$:

$$\mathcal{E}(F,F) = \int_{P_{x,y}(M)} |D_0 F(\gamma)|^2 d\nu^\lambda_{x,y}. \tag{1.1}$$

Note that the H-derivative is defined by using the Levi-Civita connection. Let $-L_\lambda$ be the non-negative generator of the above Dirichlet form. This is a generalization of the usual Ornstein-Uhlenbeck operator in the case where M is a Euclidean space. We are interested in the spectral properties of the operator $-L_\lambda$. For example, are there eigenvalues?, are there spectral gaps, where the essential spectrum is ?, and so on. In particular, the limit $\lambda \to \infty$ is a kind of semi-classical limit since formally the Brownian motion measure is written as

$$d\nu^\lambda_{x,y}(\gamma) = \frac{1}{Z_\lambda} \exp\left(-\frac{\lambda}{2}\int_0^1 |\gamma'(t)|^2 dt\right) d\gamma.$$

Therefore one may expect that the asymptotic behavior of the low-lying spectrum of $-L_\lambda$ is related with the critical points of the energy function $E(\gamma) = \frac{1}{2}\int_0^1 |\gamma'(t)|^2 dt$, that is, the set of geodesics. However, of course, this naive observation is not true in general since the space of paths $P_{x,y}(M)$ is non-compact and unbounded set and we need to put some assumptions at the infinity. Hence we study the Ornstein-Uhlenbeck operators on a certain domain with Dirichlet boundary condition in this paper which still captures essential points of the problem. We explain what kind of subsets of $P_{x,y}(M)$ we are interested in. Let $\mathcal{D} = B_r(y) = \{z \in M \mid d(z,y) < r\}$, where d stands for the Riemannian distance function on (M,g). We assume that $x \in \mathcal{D}$ and the closure of $\mathcal{D}$ does not contain the cut-locus of y. Actually we put stronger assumptions on $\mathcal{D}$ which we will explain in the next section. Let $P_{x,y}(\mathcal{D})$ be the all paths $\gamma \in P_{x,y}(M)$ satisfying $\gamma(t) \in \mathcal{D}$ $(0 \le t \le 1)$. Let $c_{x,y} = c_{x,y}(t)$ $(0 \le t \le 1)$ be the uniquely defined minimal geodesic between x and y. Then $c_{x,y} \in P_{x,y}(\mathcal{D})$ and the set $P_{x,y}(\mathcal{D})$ is an open neighborhood of $c_{x,y}$. We study the Ornstein-Uhlenbeck operator $-L_\lambda$ with Dirichlet boundary condition (we call just it a Dirichlet Laplacian) on $L^2(P_{x,y}(\mathcal{D}), d\bar{\nu}^\lambda_{x,y})$, where $\bar{\nu}^\lambda_{x,y}$ is the normalized probability measure of the restriction of $\nu^\lambda_{x,y}$ to $P_{x,y}(\mathcal{D})$. It is well-known that the pinned measure converges weakly to the atomic measure $\delta_{c_{x,y}}$ at $c_{x,y}$. This implies that the bottom of spectrum of the Dirichlet Laplacian converges to 0. The aim of this paper is to show that the second generalized lowest eigenvalue divided by λ converges to the bottom of the spectrum of the Hessian of the energy function $E(\gamma)$ at $c_{x,y}$ as $\lambda \to \infty$ which is naturally conjectured by an analogy of semi-classical analysis in finite dimensional cases. We note that a rough lower and upper bound was already given by Eberle [10]. See also [8, 9]. Also I gave a rough lower bound by using a Clark-Ocone-Haussman (=COH) formula in [3]. Our result is a refinement of such results and the proof is based on the COH formula.

The paper is organized as follows. In Section 2, we explain the assumptions on the Riemannian manifold and the domain $\mathcal{D}$. Next we give the definitions of Dirichlet Laplacian , the second generalized lowest eigenvalue and we state our main theorem. Also, we recall our COH formula. In Section 3, we identify the semi-classical limit of the coefficient operator $A(\gamma)_\lambda$ in the COH formula. To this end, the Jacobi fields along the geodesic $c_{x,y}$ play important roles. In particular we show that the limit of $A(\gamma)_\lambda$ is related with the square root of the Hessian of the energy function of E at $c_{x,y}$. This observation is important in our proof. After these preparations, we prove our main theorem in Section 4.

2 Statement of results

We assume that the Ricci curvature of (M,g) is bounded which implies the non-explosion of the Brownian motion. Actually, our problem is local one and this assumption is not important for our problem. We mention this local property later.

The Sobolev space $H^{1,2}(P_{x,y}(M), \nu^\lambda_{x,y})$ is the completion of the vector spaces of smooth cylindrical functions by the H^1-Sobolev norm:

$$\|F\|_{H^1} = \left(\|F\|^2_{L^2(\nu^\lambda_{x,y})} + \mathcal{E}(F,F)\right)^{1/2},$$

where $\mathcal{E}$ is the Dirichlet form which is defined in the introduction. As already defined, let $\mathcal{D}$ be a metric open ball centered at y with radius r. Throughout this paper, we always assume the following.

Assumption 1. (1) *Let us denote the set of cut-locus of y by* $\mathrm{Cut}(y)$*. Then there are no intersection of the closure of $\mathcal{D}$ and* $\mathrm{Cut}(y)$*. Also $x \in \mathcal{D}$.*
(2) *The Hessian of $k(z) = \frac{1}{2}d(z,y)^2$ satisfies that* $\inf_{z\in\mathcal{D}} \nabla^2 k(z) > 1/2$.

Since $\nabla_z^2 k(z)|_{z=y} = I_{T_yM}$, the above assumption (1), (2) hold in a small domain containing y. Under the above assumptions, clearly the minimal geodesic $c_{x,y} = c_{x,y}(t)$ $(0 \le t \le 1)$ $(c_{x,y}(0) = x, c_{x,y}(1) = y)$ belong to $\mathcal{D}$. Let

$$P_{x,y}(\mathcal{D}) = \{\gamma \in P_{x,y}(M) \mid \gamma(t) \in \mathcal{D} \text{ for all } 0 \le t \le 1\} \tag{2.1}$$

which is an open neighborhood of $c_{x,y}$ in $P_{x,y}(M)$. By the assumption, there are no geodesics other than $c_{x,y}$ in $P_{x,y}(\mathcal{D})$. Let

$$H_0^{1,2}(P_{x,y}(\mathcal{D}), \nu_{x,y}^\lambda) = \left\{F \in H^{1,2}(P_{x,y}(M), \nu_{x,y}^\lambda) \mid F = 0 \text{ outside } P_{x,y}(\mathcal{D})\right\} \tag{2.2}$$

which is a closed linear subspace of $H^{1,2}(P_{x,y}(M), \nu_{x,y}^\lambda)$. We denote the normalized probability $d\nu_{x,y}^\lambda/\nu_{x,y}^\lambda(P_{x,y}(\mathcal{D}))$ on $P_{x,y}(\mathcal{D})$ by $d\bar{\nu}_{x,y}^\lambda$. The non-positive generator L_λ corresponding to the densely defined closed form

$$\mathcal{E}(F,F), \quad F \in H_0^{1,2}(P_{x,y}(\mathcal{D}), \bar{\nu}_{x,y}^\lambda)$$

in the Hilbert space $L^2(P_{x,y}(\mathcal{D}), \bar{\nu}_{x,y}^\lambda)$ is the Dirichlet Laplacian on $P_{x,y}(\mathcal{D})$. Let

$$e_{Dir,1,P_{x,y}(\mathcal{D})}^\lambda = \inf_{F(\neq 0)\in H_0^{1,2}(P_{x,y}(\mathcal{D}))} \frac{\int_{P_{x,y}(\mathcal{D})} |D_0F|^2 d\bar{\nu}_{x,y}^\lambda}{\|F\|_{L^2(\bar{\nu}_{x,y}^\lambda)}^2}. \tag{2.3}$$

This is equal to $\inf \sigma(-L_\lambda)$, where $\sigma(-L_\lambda)$ denotes the spectral set of $-L_\lambda$. Next we introduce

$$\begin{aligned} &e_{Dir,2,P_{x,y}(\mathcal{D})}^\lambda \\ &= \sup_{G(\neq 0)\in L^2(\bar{\nu}_{x,y}^\lambda)} \inf\left\{ \frac{\int_{P_{x,y}(\mathcal{D})} |D_0F|^2 d\bar{\nu}_{x,y}^\lambda}{\|F\|_{L^2(\bar{\nu}_{x,y}^\lambda)}^2} \;\middle|\; F \in H_0^{1,2}(P_{x,y}(\mathcal{D})), \right. \\ &\qquad \left. (F,G)_{L^2(\bar{\nu}_{x,y}^\lambda)} = 0 \right\} \end{aligned} \tag{2.4}$$

which is the generalized second lowest eigenvalue of $-L_\lambda$. Let $P_{x,y}(M)^{H^1}$ be the subset of $P_{x,y}(M)$ consisting of H^1-paths. Let us consider the energy function of H^1-path:

$$E(\gamma) = \frac{1}{2}\int_0^1 |\gamma'(t)|^2 dt \qquad \gamma \in P_{x,y}(M)^{H^1}. \tag{2.5}$$

We use the same notation D_0 for the H-derivative of the smooth function on $P_{x,y}(M)^{H^1}$. The following is our main theorem.

Theorem 2. *We have*

$$\lim_{\lambda\to\infty} \frac{e_{Dir,2,P_{x,y}(\mathcal{D})}^\lambda}{\lambda} = e_0, \tag{2.6}$$

where $e_0 = \inf \sigma((D_0^2 E)(c_{x,y}))$.

Remark 3. *If the sectional curvature on each points of the geodesic $c_{x,y}$ is positive, then $\inf \sigma(D_0^2 E(c_{x,y})) < 1$ and the bottom of the spectrum is an eigenvalue and is not an essential spectrum. While the curvature is strictly negative, $\inf \sigma(D_0^2 E(c_{x,y})) = 1$ and 1 is not an eigenvalue and belongs to essential spectrum. This suggests that the second lowest eigenvalue, or more generally, some low-lying spectrum of the Dirichlet Laplacian on $P_{x,y}(\mathcal{D})$ over a positively curved manifold belongs to the discrete spectrum. Also if some isometry group acts M with the fixed points x and y, we may expect the discrete spectrum have some multiplicities. We show these kind of results in the case where M is a compact Lie group in a forthcoming paper. As for general Riemannian manifold cases, we need more works.*

In the proof of Theorem 2, we use a short time behavior of logarithmic derivative of heat kernels which is due to Malliavin and Stroock.

Theorem 4 (Malliavin-Stroock [12])**.** *Assume that M is compact and let $z \in \mathrm{Cut}(y)^c$. Then*

$$\lim_{t\to 0} t\nabla_z^2 \log p(t,y,z) = -\nabla_z^2 k(z) \tag{2.7}$$

uniformly on any compact subset of $\mathrm{Cut}(y)^c$.

Clearly, the same result holds on $\mathbb{R}^n$ with a Riemannian metric which coincides with the Euclidean metric outside a certain compact subset. Of course, similar result might hold in more general non-compact cases but the result for the perturbation of the Euclidean metric is enough for our purpose because we are concerned with Dirichlet Laplacian on a small domain. We explain this meaning more precisely. We already consider a metric ball $B_r(y)$ which includes x in M. In addition to (M,g), let (M',g') be another Riemannian manifold and x', y' be points on M'. Let d' be the distance function on (M',g'). Let r_* be a positive number which is greater than r. We denote by $B_{r_*}(y')$ the metric ball centered at y' with radius r_* in M'. Assume that $B_{r_*}(y')$ is diffeomorphic to an open Euclidean ball. Also assume that $x' \in B_{r_*}(y')$ and there exists an bijective Riemannian isometry Φ from $B_{r_*}(y)$ to $B_{r_*}(y')$ such that $\Phi(x) = x'$ and $\Phi(y) = y'$. Let us define $P_{x',y'}(B_r(y'))$ and the normalized probability measure $\bar{\nu}^{\lambda}_{x',y'}$ on $P_{x',y'}(B_r(y'))$ in a similar manner to $P_{x,y}(B_r(y))$ and so on. Define a mapping Ψ from $P_{x,y}(B_{r_*}(y))$ to $P_{x',y'}(B_{r_*}(y')))$ by $(\Psi\gamma)(t) = \Phi(\gamma(t)) \quad \gamma \in P_{x,y}(B_{r_*}(y))$. Note that the image of $P_{x,y}(B_r(y))$ by Ψ is exactly $P_{x',y'}(B_r(y'))$. Then using the uniqueness of the solution of the stochastic differential equation, we see that

(1) $\Psi : (P_{x,y}(B_r(y)), \bar{\nu}^{\lambda}_{x,y}) \to (P_{x',y'}(B_r(y')), \bar{\nu}^{\lambda}_{x',y'})$ and its inverse map are measure preserving map.

(2) Let $F \in H^{1,2}(P_{x',y'}(M'))$. If $F \in H^{1,2}_0(P_{x',y'}(B_r(y')))$, then

$$\tilde{F}(\gamma) := (F\circ\Psi)(\gamma)\chi(\sup_{0\le t\le 1} d'(\Psi(\gamma)(t), y)) \in H^{1,2}_0(P_{x,y}(B_r(y))),$$

where $\chi = \chi(t)$ is a non-negative smooth function such that $\chi(t) = 1$ for $t \le \frac{r+r_*}{2}$ and $\chi(t) = 0$ for $t \ge \frac{r+2r_*}{3}$. Moreover $\|F\|_{L^2(\bar{\nu}^{\lambda}_{x',y'})} = \|\tilde{F}\|_{L^2(\bar{\nu}^{\lambda}_{x,y})}$ and $\|D_0 F\|_{L^2(\bar{\nu}^{\lambda}_{x',y'})} = \|D_0\tilde{F}\|_{L^2(\bar{\nu}^{\lambda}_{x,y})}$.

The above observation implies that

$$e^{\lambda}_{Dir,2,P_{x,y}(B_r(y))} = e^{\lambda}_{Dir,2,P_{x',y'}(B_r(y'))}.$$

Hence, there are some freedom of varying the Riemannian metric outside a certain compact subset to study our problem. Hence, we may assume that M is diffeomorphic to $\mathbb{R}^n$ and the Riemannian metric is flat outside a certain bounded subset and the short time asymptotics in (2.7) holds. The key ingredient of the proof of Theorem 2 is a version of the Clark-Ocone-Haussman formula in [3] which can be extended to the above non-compact $\mathbb{R}^n$ case with a nice Riemannian metric. Since the formula is strongly related with the heat kernel $p(t,x,y)$ on M itself, the above observation is important.

Let us recall the COH formula in [3]. See also [5], [11], [1]. Let $\mathfrak{F}_t=\sigma(\{\gamma(s)\mid 0\le s\le t\})$. Let $\tau(\gamma)_t : T_xM\to T_{\gamma(t)}M$ be the stochastic parallel translation along the semi-martingale $\gamma(t)$ under ν_x^{λ} which is defined by the Levi-Civita connection. Then $b(t)=\int_0^t\tau(\gamma)_s^{-1}\circ d\gamma(s)$ is an $\mathfrak{F}_t$-Brownian motion with the covariance $E^{\nu_x^{\lambda}}[(b(t),u)(b(s),v)]=(u,v)\frac{t\wedge s}{\lambda}\quad(u,v\in T_xM)$ on T_xM under ν_x^{λ}. We recall the notion of the trivialization. Let $T\in\Gamma((\otimes^pTM)\otimes(\otimes^qT^*M))$ be a (p,q)-tensor on M. Using the (stochastic)-parallel translation $\tau(\gamma)_t : T_xM\to T_{\gamma(t)}M$, $T(\gamma(t))\in(\otimes^pT_{\gamma(t)}M)\otimes(\otimes^qT^*_{\gamma(t)}M)$ can be naturally identified with a tensor in $\otimes^pT_xM\otimes^q T_x^*M$ which we denote by $\overline{T(\gamma)}_t$. For example, for the time dependent vector field $V_y^{\lambda}(t,z)=\mathrm{grad}_z\log p\left(\frac{1-t}{\lambda},y,z\right)\quad(0\le t<1)$ we denote

$$\overline{V_y^{\lambda}(t,\gamma)}_t=\tau(\gamma)_t^{-1}V_y^{\lambda}(t,\gamma(t))\in T_xM.$$

Also we denote $\overline{\nabla V_y^{\lambda}(t,\gamma)}_t=\tau(\gamma)_t^{-1}\nabla_zV_y^{\lambda}(t,z)|_{z=\gamma(t)}$. More explicitly,

$$\overline{\nabla V_y^{\lambda}(t,\gamma)}_t=\tau(\gamma)_t^{-1}\nabla_z\mathrm{grad}_z\log p\left(\frac{1-t}{\lambda},y,z\right)\Bigg|_{z=\gamma(t)}\tau(\gamma)_t. \tag{2.8}$$

Let $w(t)=b(t)-\frac{1}{\lambda}\int_0^t\overline{V_y^{\lambda}(s,\gamma)}_s ds$. This process is defined for $t<1$ and it is not difficult to check that this can be extended continuously up to $t=1$. Let $\mathcal{N}^{x,y,t}$ be the set of all null sets of $\nu_{x,y}|_{\mathfrak{F}_t}$ and set $\mathfrak{G}_t=\mathfrak{F}_t\vee\mathcal{N}^{x,y,1}$. Then w is an $\mathfrak{G}_t$-adapted Brownian motion for $0\le t\le 1$ such that $E^{\nu_{x,y}^{\lambda}}[(w(t),u)_{T_xM}(w(s),v)_{T_xM}]=\frac{t\wedge s}{\lambda}(u,v)$ for any $u,v\in T_xM$. Let

$$K(\gamma)_{\lambda,t}=-\frac{1}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t+\frac{1}{\lambda}\overline{\nabla V_y^{\lambda}(t,\gamma)}_t. \tag{2.9}$$

Let $M(\gamma)_{\lambda,t}$ be the linear mapping on T_xM satisfying the differential equation:

$$M(\gamma)'_{\lambda,t} = K(\gamma)_{\lambda,t}M(\gamma)_{\lambda,t}\quad 0\le t\le 1, \tag{2.10}$$

$$M(\gamma)_{\lambda,0} = I. \tag{2.11}$$

Using M and K, we define

$$J(\gamma)_{\lambda}\varphi(t) = (M(\gamma)^*_{\lambda,t})^{-1}\int_t^1 M(\gamma)^*_{\lambda,s}K(\gamma)_{\lambda,s}\varphi(s)ds.$$

Then $J(\gamma)_{\lambda}$ is a bounded linear operator on L^2 for every γ. Also let

$$A(\gamma)_{\lambda}=I_{L^2}+J(\gamma)_{\lambda}. \tag{2.12}$$

Now, we state our Clark-Ocone-Haussman formula for $F\in H_0^{1,2}(P_{x,y}(\mathcal{D}))$ and its immediate consequences. We refer the readers to [6] [7] for further related developments in analysis in path spaces.

Theorem 5.
(1) *Let*

$$\xi_\lambda = \operatorname{esssup}\left\{\|A(\gamma)_\lambda\|_{op} \mid \gamma \in P_{x,y}(\mathcal{D})\right\}. \tag{2.13}$$

Then there exists $\lambda_0 > 0$ *such that* $\sup_{\lambda \geq \lambda_0} \xi_\lambda < \infty$.
(2) *For any* $F \in H_0^{1,2}(P_{x,y}(\mathcal{D}))$, $D_0F(\gamma) = 0$ *for* $\nu_{x,y}^\lambda$*-almost all* $\gamma \in P_{x,y}(\mathcal{D})^c$.
(3) *Let* $\lambda \geq \lambda_0$. *For any* $F \in H_0^{1,2}(P_{x,y}(\mathcal{D}))$, *the following COH formula holds:*

$$E^{\nu_{x,y}^\lambda}[F|\mathfrak{G}_t] = E^{\nu_{x,y}^\lambda}[F] + \int_0^t (H(s,\gamma), dw(s)), \quad 0 \leq t \leq 1,$$

where

$$H(s,\gamma) = E^{\nu_{x,y}^\lambda}\left[A(\gamma)_\lambda (D_0F(\gamma)')(s)|\mathfrak{G}_s\right]. \tag{2.14}$$

(2.14) denotes the predictable projection and $D_0F(\gamma)'_t = \frac{d}{dt}(D_0F)(\gamma)_t$.
(4) *The following inequalities hold.*

$$E^{\bar{\nu}_{x,y}^\lambda}\left[F^2 \log\left(F^2/\|F\|^2_{L^2(\bar{\nu}_{x,y}^\lambda)}\right)\right] \leq \frac{2\xi_\lambda}{\lambda} E^{\bar{\nu}_{x,y}^\lambda}\left[|D_0F|^2\right], \tag{2.15}$$

$$\frac{\lambda}{\xi_\lambda} E^{\bar{\nu}_{x,y}^\lambda}\left[\left(F - E^{\nu_{x,y}^\lambda}[F]\right)^2\right] \leq E^{\bar{\nu}_{x,y}^\lambda}\left[|D_0F|^2\right]. \tag{2.16}$$

(2.16) implies that $e^\lambda_{Dir,2,P_{x,y}(\mathcal{D})}$ diverges to $+\infty$ at least of the order of $O(\lambda)$. Therefore, $e^\lambda_{Dir,1,P_{x,y}(\mathcal{D})}$ is a simple eigenvalue. We denote the normalized non-negative eigenfunction (ground state function) by Ω. It is clear that $\Omega \in H_0^{1,2}(P_{x,y}(\mathcal{D}), \bar{\nu}_{x,y}^\lambda)$. Also we have

$$\begin{aligned} e^\lambda_{Dir,2,P_{x,y}(\mathcal{D})} &= \inf\Bigg\{ \frac{\int_{P_{x,y}(\mathcal{D})} |D_0(F-(\Omega,F)\Omega)|^2 d\bar{\nu}_{x,y}^\lambda}{\|F-(\Omega,F)\Omega\|^2_{L^2(\bar{\nu}_{x,y}^\lambda)}} \;\Bigg|\; F \in H_0^{1,2}(P_{x,y}(\mathcal{D})) \\ &\qquad \text{and } \|F-(\Omega,F)\Omega\|_{L^2(\bar{\nu}_{x,y}^\lambda)} \neq 0 \Bigg\}. \end{aligned} \tag{2.17}$$

It is plausible that Ω is a strictly positive for $\nu_{x,y}^\lambda$ almost all γ which follows from the positivity improving property of the corresponding L^2-semigroup. However, we do not need such a property in this paper and we do not consider such a problem.

3 Square root of Hessian of the energy function and Jacobi fields

Let ξ be the tangent vector at x such that $\exp_x(t\xi) = c_{x,y}(t)$ $(0 \leq t \leq 1)$, where $\exp_x$ stands for the exponential mapping at x. Clearly it holds that $d(x,y) = \|\xi\|_{T_xM}$. We denote $c_{y,x}(t) = c_{x,y}(1-t)$ which is a reverse geodesic path from y to x. By the assumption that x is not in the cut-locus of y, when $\lambda \to \infty$, the pinned Brownian motion measure converges weakly to the atomic measure $\delta_{c_{x,y}}$ at $c_{x,y}$. Also by Theorem 4, when γ and $c_{x,y}$ are close and λ is large enough, $K(\gamma)_{\lambda,t}$ can be approximated with $K(t)$ where

$$K(t) = -\frac{1}{1-t}\overline{\nabla^2 k(c_{x,y})}_t. \tag{3.1}$$

Hence when $\lambda \to \infty$, the coefficient operator $I + J(\gamma)_\lambda$ in the COH formula converges to the corresponding non-random $I + J_0$ (or $(S^{-1})^*$) which is defined using the Hessian of the square of the distance function k along the geodesic $c_{x,y}$. In fact, this observation leads our main results. We refer the readers for the notations J_0 and $(S^{-1})^*$ to Lemma 7 and Lemma 10. In order to see the explicit expression of the Hessian of $k(z)$ $(z \in c_{x,y})$, we recall the notion of Jacobi fields.

Let R be the curvature tensor and define $R(t) = \overline{R(c_{x,y})_t}(\cdot, \xi)(\xi)$ which is a linear mapping on T_xM. Also we define $R^{\leftarrow}(t) = R(1-t)$. Let $v \in T_xM$ and $W(t,v)$ be the solution to the following ODE:

$$W''(t,v) + R^{\leftarrow}(t)W(t,v) = 0 \quad 0 \le t \le 1, \quad W(0,v) = 0, \; W'(0,v) = v. \tag{3.2}$$

Since $t \to W(t,v)$ is linear, by denoting the corresponding matrix by $W(t)$, we may write $W(t,v) = W(t)v$. Of course, $W(0) = 0, W'(0) = I$. By the assumption that there are no cut-locus on $\{c_{y,x}(t)\}$, $W(t)$ is invertible linear map for all $0 < t \le 1$ and $J(t,v) = W(t)W(1)^{-1}v$ is the solution to

$$J''(t,v) + R^{\leftarrow}(t)J(t,v) = 0, \quad J(0,v) = 0, \; J(1,v) = v$$

and $(\nabla^2 k(c_{y,x}))_1(v,v) = (J'(1,v),v) = (W'(1)W(1)v,v)$. Let $0 < T \le 1$. We can obtain explicit form of the Jacobi field along $c_{y,x}(t)$ $(0 \le t \le T)$ with given terminal value at T using W. Let $J_T(t,v) = W(Tt)W(T)^{-1}v$. Then $J_T(t,v)$ $0 \le t \le 1$ satisfies the Jacobi equation

$$J_T''(t) + R^{\leftarrow}(tT)T^2 J_T(t) = 0, \quad J_T(0) = 0, \; J_T(1) = v. \tag{3.3}$$

Hence $\overline{\nabla^2 k(c_{y,x})}_t(v,v) = t\left(W'(t)W(t)^{-1}v, v\right)$. Therefore the Hessian of k at $c_{y,x}(t)$ is given by

$$\overline{\nabla^2 k(c_{y,x})}_t = tW'(t)W(t)^{-1}.$$

This can be checked by the following argument. It suffices to show that $A(t) = tW'(t)W(t)^{-1}$ is a symmetric operator for $0 < t \le 1$. We have

$$\begin{aligned} A'(t) &= W'(t)W(t)^{-1} + tW''(t)W(t)^{-1} - tW'(t)W(t)^{-1}W'(t)W(t)^{-1} \\ &= -tR^{\leftarrow}(t) - \frac{A(t)^2}{t} + \frac{A(t)}{t}. \end{aligned} \tag{3.4}$$

Note that $\lim_{t\to 0} A(t) = I$. If we extend $A = A(t)$ by setting $A(0) = I$, then $A(t)$ is continuously differentiable on $[0,1]$ and $A'(0) = 0$. One can show this by the equation of $W(t)$. Let $B(t) = A(t) - A(t)^*$, where $A(t)^*$ denotes the adjoint operator. Since $R^{\leftarrow}(t)$ is a symmetric matrix, $B(t)$ satisfies

$$B(t) = \frac{1}{t}\int_0^t (I - A(s)^*)B(s)ds + \frac{1}{t}\int_0^t B(s)(I - A(s))ds. \tag{3.5}$$

Noting that

$$\begin{aligned} &\frac{1}{t}\int_0^t (I - A(s)^*)B(s)ds \\ &\quad = \frac{I - A(t)^*}{t}\int_0^t B(s)ds - \frac{1}{t}\int_0^t (A(s)^*)'\left(\int_0^s B(r)dr\right)ds \end{aligned} \tag{3.6}$$

and using Gronwall's inequality, we obtain $B(t) = 0$ for all t which we want to show. Let $f(t) = W(1-t)$. Then f satisfies

$$f''(t) + R(t)f(t) = 0 \quad 0 \le t \le 1, \quad f(1) = 0, f'(1) = I \tag{3.7}$$

and we have $\overline{\nabla^2 k(c_{x,y})}_t = -(1-t)f'(t)f(t)^{-1}$ and $K(t) = -\frac{1}{1-t}\overline{\nabla^2 k(c_{x,y})}_t = f'(t)f(t)^{-1}$. Clearly $f(t)$ has the expansion around $t = 1$,

$$f(t) = (1-t)I + \frac{1}{2}(1-t)^2 R(1) + (1-t)^2 f_2(t),$$

where $f_2(t)$ is a matrix-valued smooth mapping. Therefore, when t is close to 1,

$$\begin{aligned} \overline{\nabla^2 k(c_{x,y})}_t &= \left(I + (1-t)R(1) + 2(1-t)f_2(t) - (1-t)^2 f_2'(t)\right) \\ &\quad \times \left(I + \frac{1}{2}(1-t)R(1) + (1-t)f_2(t)\right)^{-1}. \end{aligned} \tag{3.8}$$

Let

$$\tilde{K}(t) = K(t) + \frac{1}{1-t}. \tag{3.9}$$

Using (3.8), we see that $\tilde{K}(t)$ $(0 \le t \le 1)$ is a matrix-valued smooth mapping. Let $M(t)$ be the solution to

$$M'(t) = K(t)M(t), \ M(0) = I.$$

Let $N(t)$ be the solution to

$$N'(t) = \tilde{K}(t)N(t), \ N(0) = I.$$

Then $M(t) = (1-t)N(t)$. Note that $\sup_t(\|N(t)\|_{op} + \|N^{-1}(t)\|_{op}) < \infty$. Also we have $M(t) = f(t)f(0)^{-1}$. Let

$$L_0^2([0,1] \to T_xM) := L^2\left\{[0,1] \to T_xM \;\middle|\; \int_0^1 \varphi(t)dt = 0\right\}.$$

We may denote this set by L_0^2 for simplicity. Then $(U\varphi)(t) = \int_0^t \varphi(s)ds$ is a bijective linear isometry from $L_0^2([0,1] \to T_xM)$ to $H_0^1([0,1] \to T_xM)$. Also $U^{-1}h(t) = \dot{h}(t)$. Let us introduce an operator

$$\begin{aligned} (S\varphi)(t) &= \varphi(t) - f'(t)f(t)^{-1}\int_0^t \varphi(s)ds, && (3.10) \\ \mathrm{D}(S) &= L_0^2. && (3.11) \end{aligned}$$

By Hardy's inequality, S is a bounded linear operator from L_0^2 to L^2. Also we have the following lemma.

Lemma 6. *For any $\varphi \in L_0^2$,*

$$\|S\varphi\|^2 = ((I+T)\varphi, \varphi), \tag{3.12}$$

where I denotes the identity operator on L_0^2 and

$$T\varphi(t) = -\int_t^1 R(s)\int_0^s \varphi(u)duds + \int_0^1\int_t^1 R(s)\int_0^s \varphi(u)dudsdt. \tag{3.13}$$

Also we have

$$(D_0^2 E)(c_{x,y}) = U\,(I+T)\,U^{-1}, \tag{3.14}$$

where E is the energy function of the path (2.5).

Proof. We calculate $\|S\varphi\|^2$. Using $\lim_{t\to 1}\frac{1}{1-t}|\int_t^1 \varphi(s)ds|^2 = 0$ for any $\varphi \in L^2$, $f''(t) = -R(t)f(t)$ and $(f(t)^{-1})' = -f(t)^{-1}f'(t)f(t)^{-1}$, we have

$$\begin{aligned}
\|S\varphi\|^2 &= \|\varphi\|^2 - 2\int_0^1 \left(f'(t)f(t)^{-1}\int_0^t \varphi(s)ds, \varphi(t)\right) dt \\
&\quad + \int_0^1 \left|f'(t)f(t)^{-1}\int_0^t \varphi(s)ds\right|^2 dt \\
&= \|\varphi\|^2 + \int_0^1 \left(\left(f'(t)f(t)^{-1}\right)' \int_0^t \varphi(s)ds, \int_0^t \varphi(s)ds\right) dt \\
&\quad + \int_0^1 \left|f'(t)f(t)^{-1}\int_0^t \varphi(s)ds\right|^2 dt \\
&= \|\varphi\|^2 - \int_0^1 \left(R(t)\int_0^t \varphi(s)ds, \int_0^t \varphi(s)ds\right) dt \\
&= ((I+T)\varphi, \varphi)\,.
\end{aligned} \tag{3.15}$$

Finally, it is well-known that $(D_0^2E)(c_{x,y})(U\varphi, U\varphi)$ is equal to $((I+T)\varphi,\varphi)$. □

Let

$$(S_2\varphi)(t) = \varphi(t) + f'(t)\int_0^t f(s)^{-1}\varphi(s)ds. \tag{3.16}$$

Then again by Hardy's inequality S_2 is a bounded linear operator on L^2. Moreover it is easy to see that $\mathrm{Image}(S_2) \subset L_0^2$, $SS_2 = I_{L^2}$ and $S_2S = I_{L_0^2}$. Therefore, $S_2 = S^{-1}$ and $\mathrm{Image}(S) = L^2$. Moreover we have $S^*S = I+T$ by (3.12). Note that by identifying the dual space of a Hilbert space with the Hilbert space itself using Riesz's theorem, we view $S^* : (L^2)^* \to (L_0^2)^*$ as the operator from L^2 to L_0^2. We have the following explicit expression of S^{-1}, S^* and $(S^{-1})^*$.

Lemma 7. (1) *$S^{-1} : L^2 \to L_0^2$, $S^* : L^2 \to L_0^2$ are bijective linear isometries and we have for any $\varphi \in L^2$,*

$$\left(S^{-1}\varphi\right)(t) = \varphi(t) + f'(t)\int_0^t f(s)^{-1}\varphi(s)ds \tag{3.17}$$

$$\begin{aligned}
\left(S^{*}\varphi\right)(t) &= \varphi(t) - \int_0^1 \varphi(t)dt + \int_0^t f'(s)f(s)^{-1}\varphi(s)ds \\
&\quad - \int_0^1 \left(\int_0^t f'(s)f(s)^{-1}\varphi(s)ds\right) dt.
\end{aligned} \tag{3.18}$$

(2) *$(S^{-1})^*$ is a bijective linear isometry from L_0^2 to L^2. If we define $(S^{-1})^*$ is equal to 0 on the subset of constant functions, then for any $\varphi \in L^2$,*

$$\left((S^{-1})^*\varphi\right)(t) = \varphi(t) + (f(t)^*)^{-1}\int_t^1 f(s)^* f'(s) f(s)^{-1}\varphi(s)ds. \tag{3.19}$$

Also $(S^{-1})^*\varphi$ *can be written using* $M(t)$ *and* $K(t)$ *as*

$$\left((S^{-1})^*\varphi\right)(t) = \varphi(t) + (M(t)^*)^{-1}\int_t^1 M(s)^*K(s)\varphi(s)ds. \tag{3.20}$$

Proof. All the calculation are almost similar and so we show how to calculate $(S^{-1})^*$ only. Using $(f'(t)f(t)^{-1})^* = f'(t)f(t)^{-1}$, we have for $\varphi \in L^2$ with $supp\ \varphi \subset (0,1)$ and $\psi \in L^2$,

$$\begin{aligned}
&\left(S^{-1}\varphi, \psi\right)_{L^2} \\
&\quad = (\varphi,\psi) - \int_0^1 \left(\int_0^t f(s)^{-1}\varphi(s)ds, \left(\int_t^1 f(s)^*f'(s)f(s)^{-1}\psi(s)ds\right)'\right)dt \\
&= (\varphi,\psi) + \int_0^1 \left(\varphi(t), \left(f(t)^{-1}\right)^* \int_t^1 f(s)^*f'(s)f(s)^{-1}\psi(s)ds\right)dt.
\end{aligned} \tag{3.21}$$

This shows (3.19). Also $(S^{-1})^*\mathrm{const} = 0$ follows from $f(1) = 0$. □

We summarize the relation between S and T in the proposition below.

Proposition 8. (1) $I + T = S^*S$, $(S^{-1})^*(I+T) = S$ *and* $(I+T)^{-1} = S^{-1}(S^{-1})^*$.
(2)

$$\inf\sigma(I+T) = \inf\left\{\|S\varphi\|^2 \mid \|\varphi\|_{L^2} = 1, \varphi \in L_0^2\right\} = \frac{1}{\|(S^{-1})^*\|_{op}^2}. \tag{3.22}$$

Proof. $I+T = S^*S$ follows from Lemma 6. $(I+T)^{-1} = S^{-1}(S^{-1})^*$ follows from $(S^{-1})^* = (S^*)^{-1}$. (2) follows from (1). □

The relation $\inf\sigma((I+T)) \le \frac{1}{\|(S^{-1})^*\|_{op}^2}$ plays important role to prove the lower bound estimate in Theorem 2, while the relation $\inf\sigma(I+T) \ge \inf\left\{\|S\varphi\|^2 \mid \|\varphi\|_{L^2} = 1, \varphi \in L_0^2\right\}$ is used to prove the upper bound estimate.

4 Proof of Main Theorem

Lemma 9. *Let* Ω *be the normalized ground state function of* $-L_\lambda$. *Then* $\|\Omega - 1\|_{L^2(P_{x,y}(M),\nu_{x,y}^\lambda)} \le Ce^{-C'\lambda}$, *where* C, C' *are positive constants.*

Proof. Let $\varphi_\delta(\gamma) = \chi(\max_{0\le t\le 1} d(\gamma(t), c_{x,y}(t)))$, where χ is a non-negative smooth function such that $\chi(u) = 1$ for $|u| \le \delta$ and $\chi(u) = 0$ for $|u| \ge 2\delta$. Here δ is a sufficiently small positive number. Then $\|\varphi_\delta\|_{L^2(\nu_{x,y}^\lambda)} \ge 1 - Ce^{-C'\lambda}$ and $\|D_0\varphi_\delta\|_{L^2(\nu_{x,y}^\lambda)} \le Ce^{-C'\lambda}$. Hence

$$e^\lambda_{Dir,1,P_{x,y}(\mathcal{D})} \le Ce^{-\lambda C'}. \tag{4.1}$$

By the COH formula,

$$\|\Omega - (\Omega,1)_{L^2(\nu_{x,y}^\lambda)}\|_{L^2(\nu_{x,y}^\lambda)} \le Ce^{-C'\lambda}.$$

This implies

$$1 - (\Omega,1)^2_{L^2(\nu_{x,y}^\lambda)} = \left(\Omega, \Omega - (\Omega,1)_{L^2(\nu_{x,y}^\lambda)}\right)_{L^2(\nu_{x,y}^\lambda)} \le Ce^{-C'\lambda}$$

which shows $\|\Omega - 1\|^2_{L^2(P_{x,y}(M),\nu_{x,y}^\lambda)} \le 2Ce^{-C'\lambda}$. □

Lemma 10. *As already defined, let* $K(t) = -\frac{\overline{\nabla^2 k(c_{x,y})_t}}{1-t}$. *Also we consider a perturbation of* $K(t)$ *such that*

$$K_\varepsilon(t) = K(t) + \frac{C_\varepsilon(t)}{(1-t)^\delta},$$

where $0 < \delta < 1$ *is a constant and* $C_\varepsilon(t)$ $(0 \le \varepsilon \le 1)$ *be a symmetric matrices valued function satisfying* $\sup_t \|C_\varepsilon(t)\| \le \varepsilon$. *Let* $M_\varepsilon(t)$ *be the solution to*

$$M_\varepsilon'(t) = K_\varepsilon(t) M_\varepsilon(t) \tag{4.2}$$

$$M_\varepsilon(0) = I \tag{4.3}$$

Define

$$J_\varepsilon \varphi(t) = (M_\varepsilon(t)^*)^{-1} \int_t^1 M_\varepsilon(s)^* K_\varepsilon(s) \varphi(s) ds. \tag{4.4}$$

Then for sufficiently small ε, *there exists a positive constant* C *which is independent of* ε *such that*

$$\|J_\varepsilon - J_0\|_{op} \le C\varepsilon. \tag{4.5}$$

Also $J_0 = (S^{-1})^*$ *holds.*

Proof. First, we recall that $K(t)$ is the sum of $-\frac{1}{1-t} I$ and $\tilde{K}(t)$ as in (3.9). Taking this into account, we rewrite

$$K_\varepsilon(t) = -\frac{1}{1-t} + \tilde{K}_\varepsilon(t),$$

where $\tilde{K}_\varepsilon(t) = \tilde{K}(t) + \frac{C_\varepsilon(t)}{(1-t)^\delta}$. Let $N_\varepsilon(t)$ be the solution to

$$N_\varepsilon'(t) = \tilde{K}_\varepsilon(t) N_\varepsilon(t), \ N_\varepsilon(0) = I. \tag{4.6}$$

Note that $\tilde{K}_0(t) = \tilde{K}(t)$ and $N_0(t) = N(t)$. Then $M_\varepsilon(t) = (1-t) N_\varepsilon(t)$. To estimate $J_\varepsilon - J_0$, we need to estimate $N_\varepsilon - N_0$. Note that

$$N_\varepsilon(t) = N_0(t) \left(I + \int_0^t N_0(s)^{-1} \frac{C_\varepsilon(s)}{(1-s)^\delta} ds \right).$$

This implies

$$\sup_t |N_\varepsilon(t) - N_0(t)| \le C\varepsilon.$$

The constant C depends δ. $N_\varepsilon(t)^{-1} - N_0(t)^{-1}$ has also similar estimates. By this estimate and Hardy's inequality, we complete the proof of (4.5). ∎

Let us apply the lemma above in the case where $K_\varepsilon(t) = K(\gamma)_{\lambda,t}$. In this case, we have

$$\begin{aligned}
&K(\gamma)_{\lambda,t} \\
&= K(t) + \frac{1}{1-t} \left(\frac{1-t}{\lambda} \overline{\nabla^2 \log p \left(\frac{1-t}{\lambda}, y, \gamma \right)_t} + \overline{\nabla^2 k(c_{x,y})_t} \right) - \frac{1}{2\lambda} \overline{\mathrm{Ric}(\gamma)_t} \\
&= K(t) + \frac{1}{1-t} \left(\frac{1-t}{\lambda} \overline{\nabla^2 \log p \left(\frac{1-t}{\lambda}, y, \gamma \right)_t} + \overline{\nabla^2 k(\gamma)_t} \right) \\
&\quad + \frac{1}{1-t} \left(\overline{\nabla^2 k(c_{x,y})_t} - \overline{\nabla^2 k(\gamma)_t} \right) - \frac{1}{2\lambda} \overline{\mathrm{Ric}(\gamma)_t}.
\end{aligned} \tag{4.7}$$

Hence

$$
\begin{aligned}
C_\varepsilon(t) &= \frac{1}{(1-t)^{1-\delta}}\left(\frac{1-t}{\lambda}\overline{\nabla^2\log p\left(\frac{1-t}{\lambda},y,\gamma\right)}_t+\overline{\nabla^2 k(\gamma)}_t\right)\\
&\quad+\frac{1}{(1-t)^{1-\delta}}\left(\overline{\nabla^2 k(c_{x,y})}_t-\overline{\nabla^2 k(\gamma)}_t\right)\\
&\quad-\frac{(1-t)^\delta}{2\lambda}\overline{\mathrm{Ric}(\gamma)}_t. \qquad (4.8)
\end{aligned}
$$

Now, we are in a position to prove our main theorem.

Proof of Theorem 2. Let $l_\xi(t)=t\xi$. Let χ be a non-negative smooth function such that $\chi(t)=1$ for $t\le 1$ and $\chi(t)=0$ for $t\ge 2$. Let $\kappa>0$ and $\chi_{1,\kappa}(\gamma)=\chi(\kappa^{-1}\|\overline{b-l_\xi}\|^{2m}_{T^2,B,2m,\theta})$ and $\chi_{2,\kappa}(\gamma)=\left(1-\chi_{1,\kappa}(\gamma)^2\right)^{1/2}$. Here $\|\ \|_{T^2,B,2m,\theta}$ $(0<\theta<1, m$ is a large positive integer) denotes the norm for the Brownian rough path $\overline{b-l_\xi}$ over $b-l_\xi$. See Definition 7.2 in [2]. Actually, we need quasi-sure version of Brownian rough path as in Theorem 3.1 in [4] because we are considering the pinned Brownian motion measure. By Lemma 7.11 in [2], there exists a positive constant C_κ such that

$$
|D_0\chi_{1,\kappa}(\gamma)|_{H^1_0}+|D_0\chi_{2,\kappa}(\gamma)|_{H^1_0}\le C_\kappa \qquad \nu^\lambda_{x,y}-a.s.\gamma. \qquad (4.9)
$$

First we prove the upper bound estimate. Let us fix a positive number $\varepsilon>0$. Let us choose $\varphi_\varepsilon\in L^2_0$ with $\|\varphi_\varepsilon\|=1$ such that

$$
\max\left(\left|\|(I+T)\varphi_\varepsilon\|-e_0\right|,\left|\|S\varphi_\varepsilon\|^2-e_0\right|\right)\le\varepsilon. \qquad (4.10)
$$

This is possible because of Lemma 6 and Proposition 8. Let $\psi_\varepsilon=U\varphi_\varepsilon$. Let $F_\varepsilon(\gamma)=\sqrt{\lambda}\,(b-l_\xi,\psi_\varepsilon)_{H^1_0}$ and $\tilde{F}_\varepsilon=F_\varepsilon\chi_{1,\kappa}$. Note that

$$
(D_0)_h b(t)=h(t)+\int_0^t\int_0^s\overline{R(\gamma)}_u(h(s),\circ db(u))(\circ db(s)).
$$

Hence

$$
\begin{aligned}
D_0F_\varepsilon(\gamma)'_t &= \sqrt{\lambda}\psi'_\varepsilon(t)+\sqrt{\lambda}\int_t^1\overline{R(\gamma)}_s\left(\int_s^1\varphi_\varepsilon(r)db^i(r),\circ db(s)\right)\varepsilon_i\\
&\quad-\sqrt{\lambda}t\int_0^1\int_t^1\overline{R(\gamma)}_s\left(\int_s^1\varphi_\varepsilon(r)db^i(r),\circ db(s)\right)\varepsilon_i dt\\
&= \sqrt{\lambda}\psi'_\varepsilon(t)-\sqrt{\lambda}\int_t^1 R(s)\left(\psi_\varepsilon(s),\xi\right)\xi ds\\
&\quad+\sqrt{\lambda}\int_0^1\int_t^1 R(s)\left(\psi_\varepsilon(s),\xi\right)\xi ds dt+I(\lambda)\\
&= \sqrt{\lambda}(I+T)(\varphi_\varepsilon)(t)+I(\lambda). \qquad (4.11)
\end{aligned}
$$

By applying the Cameron-Martin formula to the translation $b\to b+l_\xi$ and using the similar argument to page 33 in [13], we obtain $E^{\nu^\lambda_{x,y}}[|I(\lambda)|^p]\le C_{p,\kappa}$. Here the constant $C_{p,\kappa}$ is independent of λ. Also we have $E[|D_0\chi_{1,\kappa}|^2_{H^1_0}]\le Ce^{-C'\lambda}$ for some $C,C'>0$ which follows from the Schilder type large deviation principle. This implies

$$
\|D_0\tilde{F}_\varepsilon\|^2_{L^2(\nu^\lambda_{x,y})}=\lambda\|(I+T)\varphi_\varepsilon\|^2_{L^2}+o(\lambda). \qquad (4.12)
$$

Combining $\|D_0\Omega\| \le Ce^{-C'\lambda}$, we obtain

$$\|D_0\tilde{F}_\varepsilon - \left(\tilde{F}_\varepsilon, \Omega\right) D_0\Omega\|^2_{L^2(\nu^\lambda_{x,y})} = \lambda\|(I+T)\varphi_\varepsilon\|^2_{L^2} + o(\lambda). \tag{4.13}$$

On the other hand, by the continuity theorem of rough path, there exists $\varepsilon' > 0$ such that for $\|\overline{b - l_\xi}\|_{T^2,B,2m,\theta} \le \varepsilon'$,

$$\|\overline{\nabla^2 k(c_{x,y})}_t - \nabla^2 k(\gamma)_t\| \le \varepsilon|1-t|^{\theta/2}. \tag{4.14}$$

By Lemma 3.3 in [11], for any compact subset $F \subset \mathrm{Cut}(y)^c$,

$$\sup_{z\in F}\left|t\nabla^2 \log p(t,z,y) - \nabla^2 k(z)\right| \le C_F t^{1/2}. \tag{4.15}$$

Therefore, by setting κ to be sufficiently small, $C_\varepsilon(t)$ which is defined in (4.8) satisfies the assumption in Lemma 10 for certain $\delta > 1/2$. Hence, by Lemma 10, by taking κ to be sufficiently small,

$$\begin{aligned} A(\gamma)_\lambda(D_0\tilde{F}_\varepsilon(\gamma)')_t &= \left(S^{-1}\right)^*\left(D_0\tilde{F}_\varepsilon(\gamma)'\right)_t + (J(\gamma)_\lambda - J_0)(D_0\tilde{F}_\varepsilon(\gamma)')_t \\ &= \sqrt{\lambda}S\varphi_\varepsilon(t) + I_2(\lambda), \end{aligned} \tag{4.16}$$

where $|I_2(\lambda)|_{L^2([0,1])} \le C\varepsilon\sqrt{\lambda}$. Also we have used that $(S^{-1})^*(I+T) = S$. This estimate, (4.9) and the COH formula (2.14) implies that

$$\|\tilde{F}_\varepsilon - E^{\nu^\lambda_{x,y}}[\tilde{F}_\varepsilon]\|^2_{L^2(\nu^\lambda_{x,y})} = \|S\varphi_\varepsilon\|^2 + C\varepsilon. \tag{4.17}$$

Using Lemma 9,

$$\|\tilde{F}_\varepsilon - \left(\tilde{F}_\varepsilon, \Omega\right)\Omega\|^2_{L^2(\nu^\lambda_{x,y})} = \|S\varphi_\varepsilon\|^2 + C\varepsilon + Ce^{-C'\lambda}. \tag{4.18}$$

By using the estimates (4.13), (4.18) and (4.10), we complete the proof of the upper bound. We prove lower bound. Take $F \in H^{1,2}_0(P_{x,y}(\mathcal{D}))$ such that $\|F\|_{L^2(\bar{\nu}^\lambda_{x,y})} = 1$ and $(F,\Omega) = 0$. By the IMS localization formula,

$$\mathcal{E}(F,F) = \sum_{i=1,2}\mathcal{E}(F\chi_{i,\kappa}, F\chi_{i,\kappa}) - \sum_{i=1,2} E^{\bar{\nu}^\lambda_{x,y}}[|D_0\chi_{i,\kappa}|^2F^2]. \tag{4.19}$$

By Lemma 9,

$$|E^{\nu^\lambda_{x,y}}[F]| = |E^{\nu^\lambda_{x,y}}[F(1-\Omega)]| \le \|1-\Omega\| \le Ce^{-C'\lambda}. \tag{4.20}$$

Again by taking κ to be sufficiently small, by Lemma 10 and the COH formula

$$\begin{aligned} &\|F\chi_{1,\kappa} - E^{\nu^\lambda_{x,y}}[F\chi_{1,\kappa}]\|^2_{L^2(P_{x,y}(M))} \\ &\quad\le \frac{\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^2}{\lambda} E^{\nu^\lambda_{x,y}}\left[|D_0(F\chi_{1,\kappa})|^2\right]. \end{aligned} \tag{4.21}$$

Consequently, we have

$$\|F\chi_{1,\kappa}\|^2_{L^2(P_{x,y}(\mathcal{D}))} \le \frac{\left(\|(S^{-1})^*\|_{op} + C\varepsilon\right)^2}{\lambda} E^{\nu^\lambda_{x,y}}\left[|D_0(F\chi_{1,\kappa})|^2\right] + Ce^{-C'\lambda}. \tag{4.22}$$

Now we estimate the Dirichlet norm of $F\chi_{2,\kappa}$. The log-Sobolev inequality (2.15) implies that there exists a positive constant C such that for any $F \in H_0^{1,2}(P_{x,y}(\mathcal{D}))$ and bounded measurable function V,

$$\mathcal{E}(F,F) + E^{\bar{\nu}^{\lambda}_{x,y}}\left[\lambda^2 V F^2\right] \geq -\frac{\lambda}{C}\log E^{\bar{\nu}^{\lambda}_{x,y}}\left[e^{-C\lambda V}\right]\|F\|^2_{L^2(\bar{\nu}^{\lambda}_{x,y})}. \tag{4.23}$$

We apply this inequality. Take a non-negative smooth function $\tilde{\chi}$ on $\mathbb{R}$ such that $\tilde{\chi} = 1$ on a neighborhood of the support of $1-\chi^2$ and $\tilde{\chi} = 0$ on a neighborhood of 0. Let δ be a sufficiently small positive number δ and define

$$V(\gamma) = \delta\tilde{\chi}(\kappa^{-1}\|\overline{b - l_\xi}\|^{2m}_{T^2,B,2m,\theta}).$$

By (4.23), there exists $\delta' > 0$ such that

$$\begin{aligned}
&\mathcal{E}(F\chi_{2,\kappa}, F\chi_{2,\kappa})\\
&\qquad = \mathcal{E}(F\chi_{2,\kappa}, F\chi_{2,\kappa}) - \lambda^2 E^{\bar{\nu}^{\lambda}_{x,y}}\left[V(F\chi_{2,\kappa})^2\right] + \lambda^2 E^{\bar{\nu}^{\lambda}_{x,y}}\left[V(F\chi_{2,\kappa})^2\right]\\
&\geq \quad -\frac{\lambda}{C}\log E^{\bar{\nu}^{\lambda}_{x,y}}\left[e^{C\lambda V}\right]\|F\chi_{2,\kappa}\|^2 + \lambda^2\delta\|F\chi_{2,\kappa}\|^2\\
&\geq \quad -\frac{\lambda}{C}\log\left(1 + e^{-\lambda\delta'}\right)\|F\chi_{2,\kappa}\|^2 + \lambda^2\delta\|F\chi_{2,\kappa}\|^2\\
&\geq \quad (\lambda^2\delta - C\lambda e^{-\lambda\delta'})\|F\chi_{2,\kappa}\|^2.
\end{aligned} \tag{4.24}$$

By the estimates (4.9), (4.22), (4.24) and by Proposition 8, we complete the proof. ☐

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